Closing today: 3.4(1)(2)
Closing *Tues*: 10.2
Closing Fri: 3.5(1)(2)
Office Hours - 1:30-3:00 in COM B-006

<u>10.2 Parametric Equations</u> *(continued)*

Recall: Given $x = x(t), y = y(t)$, we find the slope of the tangent line using

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Entry Task: The motion of a particular pitched baseball is given by

$$x(t) = 142t$$

$$y(t) = -16t^2 + 4t + 5$$

Find the equation of the tangent line at $t = 1/2$

$$\frac{dx}{dt} = 142$$

$$\frac{dy}{dt} = -32t + 4$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(-32t+4)}{142}$$

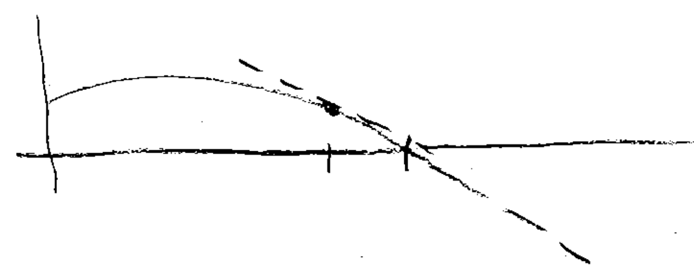
$$\left.\frac{dy}{dx}\right|_{t=1/2} = \frac{-32(1/2)+4}{142} = -\frac{12}{142} = -\frac{6}{71}$$

$$\approx -0.0845$$

$$x(1/2) = 142(1/2) = 71$$

$$y\left(\tfrac{1}{2}\right) = -16\left(\tfrac{1}{2}\right)^2 + 4\left(\tfrac{1}{2}\right) + 5 = -4 + 2 + 5 = 3$$

$$\boxed{y = -\frac{6}{71}(x - 71) + 3}$$



Example: Old test question

Find all points on

$$x(t) = t^2 + t + 3$$

$$y(t) = t^3 - 2$$

when the tangent line has slope 1

$$\frac{dy}{dx} = \frac{3t^2}{2t+1} \stackrel{?}{=} 1$$

$$\Rightarrow 3t^2 = 2t + 1$$

$$\Rightarrow 3t^2 - 2t - 1 = 0$$

$$(3t+1)(t-1) = 0$$

$$t = -1/3, \quad t = 1$$

POINTS $(x,y) = (x(-1/3), y(-1/3)) = \left(\frac{25}{9}, \frac{-55}{27}\right)$

$(x,y) = (x(1), y(1)) = (5, -1)$

Speed: For a parametric equation, it is natural to ask what the "speedometer" speed is for the moving object.

"average speed from t to $t+h$" $= \dfrac{\text{change in distance}}{\text{change in time}}$

$$\approx \frac{\sqrt{\left(x(t+h)-x(t)\right)^2+\left(y(t+h)-y(t)\right)^2}}{h}$$

$$= \sqrt{\left(\frac{x(t+h)-x(t)}{h}\right)^2 + \left(\frac{y(t+h)-y(t)}{h}\right)^2}$$

"instantaneous speed at t" is the limit of the above expressions as $h \to 0$

$$= \sqrt{(x'(t))^2 + (y'(t))^2}$$

Thus,

$$\text{speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Example: Again,

$$x(t) = 142t$$

$$y(t) = -16t^2 + 4t + 5$$

Find the speed of the ball at $t = 1/2$

$x'(\tfrac{1}{2}) = 142$

$y'(\tfrac{1}{2}) = -32(1/2) + 4 = -12$

$\text{SPEED} = \sqrt{(142)^2 + (-12)^2}$

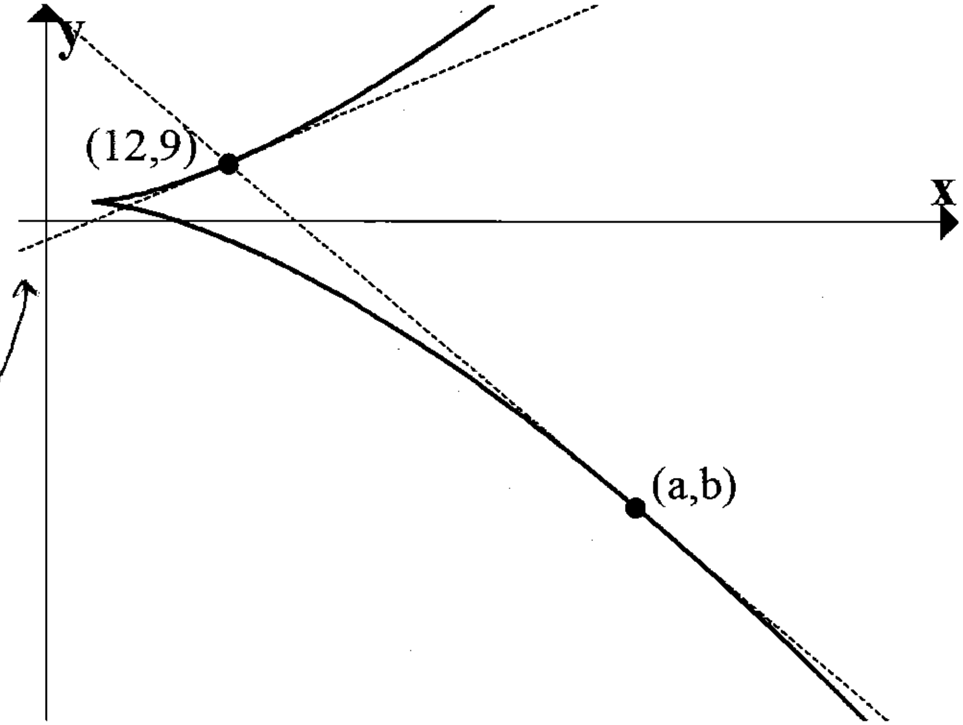
$\approx 142.506 \text{ ft/sec}$

HW10.2 #7 Hint:

$x = 9t^2 + 3, y = 6t^3 + 3$

There are two tangent lines to this curve that **also** pass through (12,9).

Find these two tangent lines.



TWO JOBS!

$\boxed{\text{I}}$ FIND WHEN $9t^2+3 \stackrel{?}{=} 12$

$6t^3+3 \stackrel{?}{=} 9$

AND FIND EQUATION FOR TANGENT

$\boxed{\text{II}}$ SOLVE FOR (a,b)

$a = 9t^2+3$

$b = 6t^3+3$

$\vdots$

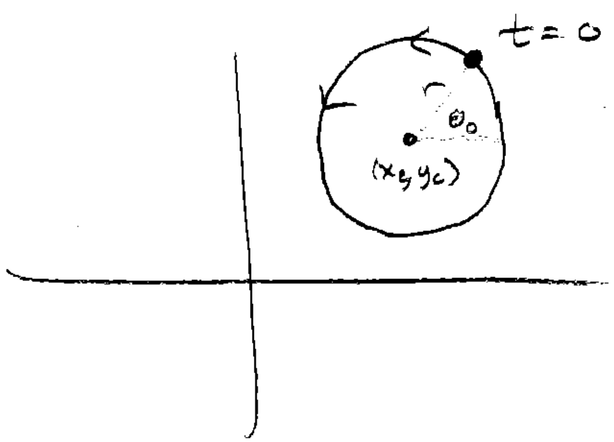
Special parametric equations:

1. Uniform Circular Motion:

$$x = x_c + r\cos(\theta_0 + \omega t)$$
$$y = y_c + r\sin(\theta_0 + \omega t)$$

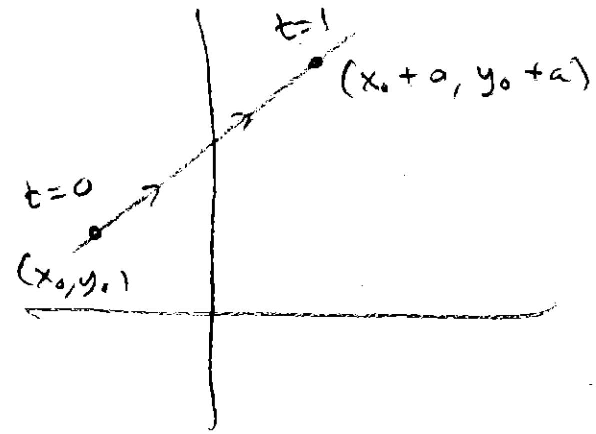
Note the fundamental circular motion facts from precalculus (*only* true in radians):

$$\text{linear speed} = v = \omega r,$$
$$\text{arc length} = s = r\theta$$

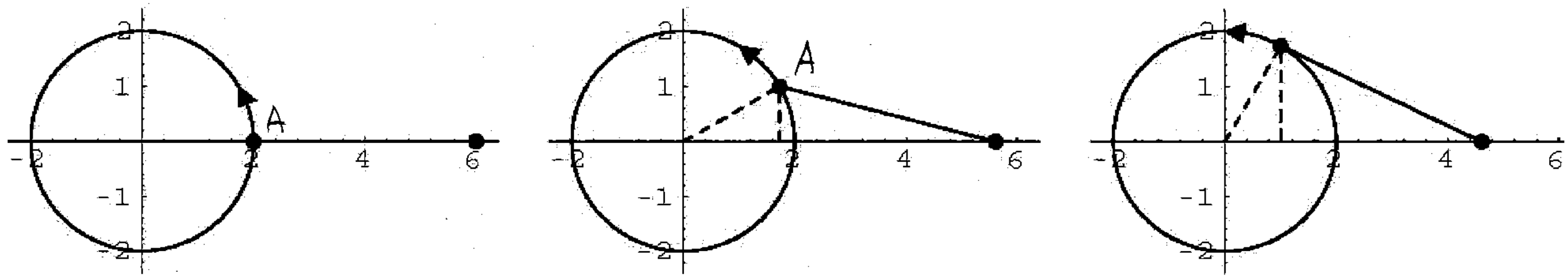


2. Uniform Linear Motion:

$$x = x_0 + at$$
$$y = y_0 + bt$$



From HW (Piston Problem): A 4cm rod is attached at one end to a point, A, on a wheel of radius 2 cm. The other end B is free to move back and forth along a horizontal bar that goes through the center of the wheel. At time $t=0$ the rod is situated as in the diagram at the left below. The wheel rotates at 3.5 rev/sec.



Find parametric equation for the point A and the point B.

$\boxed{A}$ $\theta_0 = 0$, $\omega = 3.5 \frac{\text{rev}}{\text{sec}} \cdot \frac{2\pi \text{ rad}}{1 \text{ rev}} = 7\pi \frac{\text{rad}}{\text{sec}}$

$r = 2$

$x = 2\cos(7\pi t)$

$y = 2\sin(7\pi t)$

$\boxed{B}$ DISTANCE FROM A TO B IS ALWAYS 4!

$x = ??$

$y = 0$

$$\sqrt{(x - 2\cos(7\pi t))^2 - (0 - 2\sin(7\pi t))^2} = 4$$

Solve for x!

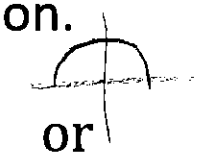
<u>3.5 Implicit Differentiation</u>

Motivation: Consider the unit circle

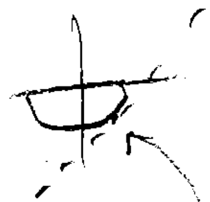
$$x^2 + y^2 = 1$$

Does NOT define a function. It *implicitly* defines more than one function.

$$y = f(x) = \sqrt{1 - x^2} \quad \text{or}$$



$$y = g(x) = -\sqrt{1 - x^2}$$



Questions:

1. Find $f'(x)$ and $g'(x)$.
2. What is the slope of the tangent line at $(x, y) = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$?

use $g(x)$

$$f(x) = (1 - x^2)^{1/2}$$

$$f'(x) = \frac{1}{2}(1 - x^2)^{-1/2} \cdot (-2x)$$

$$\boxed{f'(x) = \frac{-x}{\sqrt{1 - x^2}}}$$

$$\boxed{g'(x) = \frac{x}{\sqrt{1 - x^2}}}$$

$$g'\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}/2}{\sqrt{1 - (\sqrt{3}/2)^2}}$$

$$= \frac{\sqrt{3}/2}{\sqrt{1 - 3/4}} = \frac{\sqrt{3}/2}{1/2}$$

$$\boxed{g'\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3} \quad \text{slope!}}$$

New idea (Implicit Differentiation):

Given $x^2 + y^2 = 1.$

Think of y as a function of x and differentiate directly to save time and energy (and gain simplicity).

So think of it as:

$$\frac{d}{dx}\left[\ x^2 + (y(x))^2 = 1.\right]$$

$$2x + 2(y(x)) \cdot \frac{dy}{dx} = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = -2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2x}{2y} = -x/y$$

Again: What is the slope of the tangent line at $(x, y) = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$?

$$\frac{dy}{dx} = -x/y$$

$$= -\frac{(\sqrt{3}/2)}{(-1/2)} = \sqrt{3}$$

NOTE: $\frac{dy}{dx} = -x/y$

MATCHES __BOTH__

$f'(x)$ & $g'(x)$.

General Notes (Implicit Differentiation)

Given any equation of the form:

$$F(x, y) = 0,$$

we think of *y* as an *implicit* function of *x*

$$F(x, y(x)) = 0$$

and differentiate directly (correctly using the chain rule as we go!)

Quick Examples: Find dy/dx

output input

1. $y^2 = x$

$y = y(x)$

$$\frac{d}{dx}\left[(y(x))^2 = x\right]$$

$$2\,y(x)\,\frac{dy}{dx} = 1$$

$$2y\,\frac{dy}{dx} = 1$$

$$\boxed{\frac{dy}{dx} = \frac{1}{2y}}$$

$$y = \pm\sqrt{x} = \pm x^{1/2}$$

$$y = -x^{1/2} \qquad y = +x^{1/2}$$

$$y' = -\tfrac{1}{2}x^{-1/2} \qquad y' = \tfrac{1}{2}x^{-1/2}$$

$$y' = -\frac{1}{2\sqrt{x}} \qquad y' = \frac{1}{2\sqrt{x}}$$

2. $x^2y + y^2 = 3$

$$\frac{d}{dx}\left[x^2(y(x)) + (y(x))^2 = 3\right]$$

$$2xy + x^2\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$(x^2+2y)\frac{dy}{dx} = -2xy$$

$$\boxed{\frac{dy}{dx} = \frac{-2xy}{x^2+2y}}$$

3. $xe^y + \tan(x) + \sin(y) = 1$

$$e^y + xe^y\frac{dy}{dx} + \sec^2(x) + \cos(y)\frac{dy}{dx} = 0$$

$$(xe^y + \cos(y))\frac{dy}{dx} = -e^y - \sec^2(x)$$

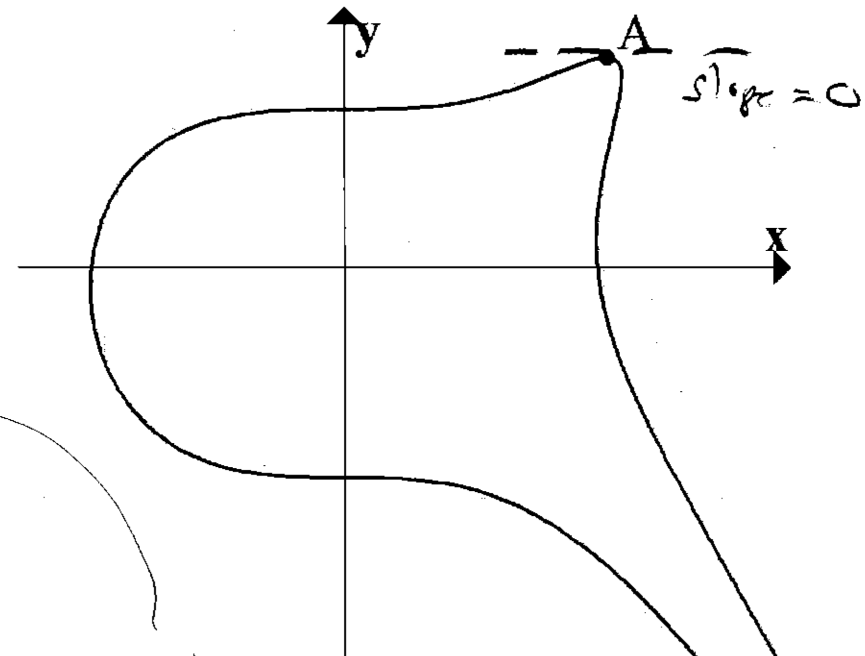
$$\boxed{\frac{dy}{dx} = \frac{-e^y - \sec^2(x)}{xe^y + \cos(y)}}$$

Old Midterm Question:

Consider the curve implicitly defined by

$$(x^3 - y^2)^2 + e^y = 4.$$

Find the (x, y) coordinates of the point *A* shown (highest point on the curve).



$$2(x^3-y^2)\cdot\left(3x^2-2y\frac{dy}{dx}\right)+e^y\frac{dy}{dx}=0$$

WANT TO FIND (x, y) WHEN $\frac{dy}{dx}=0$.

$$\Rightarrow \quad 2(x^3-y^2)\cdot(3x^2-0)+0=0$$

$$\Rightarrow \quad 6x^2(x^3-y^2)=0$$

$$\Rightarrow \quad 6x^2=0 \quad \text{or} \quad \boxed{x^3-y^2=0}$$

<u>NO</u>

<u>AND</u> WE KNOW $(x^3-y^2)^2+e^y=4$

$$\Rightarrow \quad 0^2+e^y=4 \Rightarrow \boxed{y=\ln(4)}$$

$$\rightarrow x^3-(\ln(4))^2=0$$

$$x^3=(\ln(4))^2$$

$$\boxed{x=(\ln(4))^{2/3}}$$